Math 213, Fall 2025, Final Exam Solutions
Read Carefully, Write Neatly, Sketch Accurately

1. Find the center and radius of the sphere given by: $(x+3)^2 + (y-1)^2 + (z-2)^2 = 25$
Solution:
The center is (-3, 1, 2) and the radius is 5

2. Find the vector projection of the vector $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ onto $\mathbf{v} = 3\mathbf{i} + 0\mathbf{j} + 4\mathbf{k}$.
Solution:

$$\mathrm{proj}_{\mathrm{v}}\mathrm{u} = \frac{\mathrm{u}\cdot\mathrm{v}}{|\mathrm{v}|^2}\mathrm{v} = \frac{\langle 2, -2, 1\rangle \cdot \langle 3, 0, 4\rangle}{\langle 3, 0, 4\rangle \cdot \langle 3, 0, 4\rangle}\langle 3, 0, 4\rangle = \frac{10}{25}\langle 3, 0, 4\rangle = \left\langle \frac{6}{5}; 0, \frac{8}{5} \right\rangle$$

3. Find the tangent line to the curve $\mathbf{r}(\mathbf{t}) = t^2\mathbf{i} + (2t-1)\mathbf{j} + t^2\mathbf{k}$ when t=3.
Set up the integral for the length of the curve $\mathbf{r}(t) = t^2\mathbf{i} + (2t-1)\mathbf{j} + t^2\mathbf{k}$ from t=3 to t=5.
Do not evaluate.

Solution:
The velocity vector is:
$\mathbf{r}'(\mathbf{t}) = 2t\mathbf{i} + 2\mathbf{j} + 2t\mathbf{k} \Rightarrow \mathbf{r}'(3) = 6\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ also $\mathbf{r}(3) = 9\mathbf{i} + 5\mathbf{j} + 9\mathbf{k}$
So the desired tangent line is $x = 9 + 6t; y = 5 + 2t; z = 9 + 6t$
$|\mathrm{v}(\mathrm{t})| = \sqrt{4\mathrm{t}^2 + 4 + 4\mathrm{t}^2}$ and the arclength is $\int_3^5 \sqrt{8\mathrm{t}^2 + 4}\,dt$

4. If $(y + z^3)x = 0$ defines a surface, find the equation of the tangent plane and the normal line (parametric equations) to the surface at the point (1, -8, 2).
Solution:
The normal line is in the direction of the gradient
if $G = (y + z^3)x$ then $\nabla G = (y + z^3)\mathbf{i} + x\mathbf{j} + 3z^2x\mathbf{k} \Rightarrow \nabla G(1, -8, 2) = 0\mathbf{i} + \mathbf{j} + 12\mathbf{k}$ so the normal line is $x = 1;\ y = -8 + t; z = 2 + 12t$ and the tangent plane is
$0(x-1) + (y+8) + 12(z-2) = 0$

5. Find the local maxima, minima and saddle points of $f(x, y) = x^2 - 2xy + 2y^2 - 2x + 2y + 3$ and classify them.
Solution:
Set the partial derivatives to zero to find the critical point(s) then use the second partials and the discriminant to classify them.
$f_x = 2x - 2y - 2;\ f_y = -2x + 4y + 2$

$\Rightarrow$ (setting to zero and adding the equations) $2y = 0 \Rightarrow y = 0, x = 1$

$f_{xx} = 2; f_{xy} = -2; f_{yy} = 4 \Rightarrow f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 = 8 - 4 > 0$

$\Rightarrow$ There is a local minimum at $(1, 0)$ where $f = 1 - 0 + 0 - 2 + 0 + 3 = 2$

6. Find the point(s) on the curve $x^2y^2=81$ that are closest to the origin using Lagrange multipliers.

Solutions:

Minimize distance to the origin x^2+y^2 subject to $x^2y^2=81$. Take gradient of both which must go in the same direction. This implies

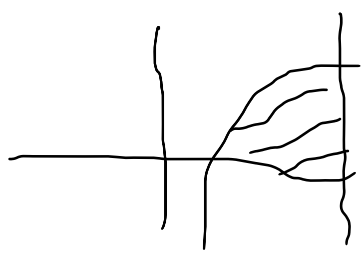
$\langle 2x,2y\rangle=\lambda\langle 2xy^2,2x^2y\rangle$ and $x^2y^2=81 \Rightarrow 2x=2xy^2\lambda;\ 2y=2x^2y\lambda$ and $x^2y^2=81$

$\Rightarrow$ (since x and y cannot be 0) $\lambda y^2=1$; and $\lambda x^2=1 \Rightarrow y=\pm x$ so $x^2y^2=81$ becomes $x^4=81$

$\Rightarrow x=\pm 3;\ y=\pm 3$. The points are $(3,3)$ $(3,-3)$, $(-3,3)$ and $(-3,-3)$ (if

7. Reverse the order of integration for the integral $\int_0^{\ln 2}\int_{e^y}^{2} dxdy$. Do not evaluate.

Solution: x goes from e^y (curve $y=\ln x$) to $x=2$ and y goes from 0 to ln 2. So we have y goes from 0 to ln x and x goes from 0 to 2. $\int_1^2\int_0^{\ln x} dydx$



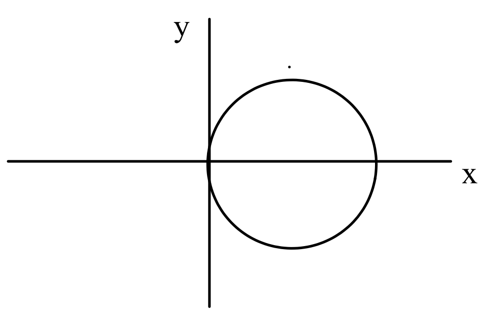
8. Give the limits of integration for evaluating the integral $\iiint_D f(r,\theta,z)\, rdzdrd\theta$ as an iterated integral over the solid region D that is bounded below by the plane $z=0$, on the side by the cylinder $r=\cos\theta$ and on the top by the paraboloid $z=3(x^2+y^2)$. Sketch the curve where the cylinder intersects the xy-plane.

Solution:

The base of the cylinder is the circle

$r=\cos\theta \Rightarrow r^2=r\cos\theta \Rightarrow x^2+y^2=x \Rightarrow x^2-x+y^2=0 \Rightarrow \left(x-\frac{1}{2}\right)^2+y^2=\frac{1}{4}$.

So the base is a circle of radius ½ with center (1/2,0).



So the region goes from 0 to $3r^2$ in z, from 0 to the cylinder $r = \cos\theta$ in r and from -pi/2 to pi/2 in theta, giving:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\cos\theta} \int_{0}^{3r^2} f(r,\theta,z)\, r dz dr d\theta$$

9. Show that the differential form $\sin y \cos x dx + \cos y \sin x dy + dz$ is exact. Find the potential function and integrate $\int_{(1,0,2)}^{(3,4,0)} \sin y \cos x dx + \cos y \sin x dy + dz$

Solution:

Show the determinant is zero $\sin y \cos x$

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \sin y \cos x & \cos y \sin x & 1 \end{vmatrix} = i(0-0) + j(0-0) + k(\cos y \cos x - \cos y \cos x) = 0$$

So $\int_{(1,0,2)}^{(3,4,0)} \sin y \cos x dx + \cos y \sin x dy + dz = \sin y \sin x + z \Big|_{(1,0,2)}^{(3,4,0)} = \sin 4 \sin 3 - 2$

10. Find directly (without using Green's Theorem) the counterclockwise circulation for the field $\vec{\mathbf{F}} = (x-2y)\,\vec{\mathbf{i}} + x\,\vec{\mathbf{j}}$ around the unit circle centered at the origin.

Solution:

$$\vec{\mathbf{F}} = (x-2y)\vec{\mathbf{i}} + x\vec{\mathbf{j}}$$

$$\vec{\mathbf{r}} = \cos t\,\vec{\mathbf{i}} + \sin t\,\vec{\mathbf{j}}$$

$$\vec{\mathbf{dr}} = -\sin t\,\vec{\mathbf{i}} + \cos t\,\vec{\mathbf{j}}$$

$$\oint \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}} = \int_0^{2\pi} \mathbf{< \cos t - 2\sin t, \cos t > \cdot < -\sin t, \cos t > dt}$$

$$= \int_0^{2\pi} (-\sin t\cos t + 2\sin^2 t + \cos^2 t)dt = \int_0^{2\pi} (-\sin t\cos t + \sin^2 t + \sin^2 t + \cos^2 t)dt$$

$$= \int_0^{2\pi} \left(-\sin t\cos t + \left(\frac{1}{2} - \frac{\cos 2t}{2}\right) + 1\right)dt = \frac{3}{2}(2\pi) = 3\pi$$

11. Provide a parametrization of the form $\mathbf{r}(r,\theta) = f(r,\theta)\,\mathbf{i} + g(r,\theta)\,\mathbf{j} + h(r,\theta)\,\mathbf{k}$ of the surface cut from the elliptical paraboloid $z = x^2 + 4y^2$ by the planes $z=0$ and $z = 4$.

Solution:

$\mathbf{r}(r,\theta) = r\cos\theta\mathbf{i} + \frac{1}{2} r \sin\theta\mathbf{j} + r^2\mathbf{k}$ where $0 \le r \le 2;\ 0 \le \theta \le 2\pi$

Other solutions are possible.

12. Find the surface area differential $d\sigma$ for the parametrization $\mathbf{r}(x, z) = z\mathbf{i} + xz\mathbf{j} + x\mathbf{k}$

Solution:

$$d\sigma = \left|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\right| dudv = \begin{vmatrix} i & j & k \\ 0 & z & 1 \\ 1 & x & 0 \end{vmatrix} dxdz = \left|-xi + j - zk\right| dxdz = \sqrt{x^2 + 1 + z^2}\,dxdz$$

13. Set up the surface integral (using the right hand side of Stokes' theorem) for the circulation of the field $\mathbf{F} = xy\mathbf{i} + z\mathbf{j} + xz\mathbf{k}$ (counterclockwise as viewed from above) around the boundary of the triangle in the first octant cut from the plane $x + 2y + 3z = 6$. Do not evaluate. (Your answer should be a double integral in terms of x and y.)

Solution:

$$\oint \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint \left(\nabla \times \vec{\mathbf{F}}\right) \cdot \vec{\mathbf{n}}\, d\sigma;$$

$$\nabla \times \vec{\mathbf{F}} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ xy & z & xz \end{vmatrix} = -i - zj - xk$$

$$\vec{\mathbf{n}} = \frac{\nabla f}{|\nabla f|} = \frac{\langle 1,2,3 \rangle}{\sqrt{14}} \text{ where } f = x + 2y + 3z - 6$$

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot k|} = \frac{\sqrt{14}}{3}$$

$$\oint \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint \left(\nabla \times \vec{\mathbf{F}}\right) \cdot \vec{\mathbf{n}}\, d\sigma = \int_0^3 \int_0^{6-2y} \langle -1, -z, -x \rangle \cdot \frac{\langle 1, 2, 3 \rangle}{\sqrt{14}} \frac{\sqrt{14}}{3} dxdy$$

$$z = \frac{(6 - x - 2y)}{3} \Rightarrow \int_0^3 \int_0^{6-2y} \left\langle -1, -\frac{(6 - x - 2y)}{3}, -x \right\rangle \cdot \frac{\langle 1, 2, 3 \rangle}{\sqrt{14}} \frac{\sqrt{14}}{3} dxdy$$

$$= \frac{1}{3} \int_0^3 \int_0^{6-2y} \left(-1 + \frac{2}{3}(6 - x - 2y) - 3x\right) dxdy$$

14. Use the Divergence Theorem to find the inward flux of the field $\mathbf{F} = \frac{x^3}{3}\mathbf{i} + \frac{y^3}{3}\mathbf{j} + 2z\mathbf{k}$ across the boundary of the region bounded by the cylinder $x^2 + y^2 \leq 4$ and the planes $z=0$ and $z=3$

Solution:

$$\iint \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, d\sigma = \int \iint \left(\nabla \cdot \vec{\mathbf{F}}\right) dV = \int \iint \left(x^2 + y^2 + 2\right) dxdydz$$

$$= \int_0^{2\pi} \int_0^2 \int_0^3 \left(r^2 + 2\right) dzrdrd\theta = 3\int_0^{2\pi} \int_0^2 \left(r^2 + 2\right) rdrd\theta = 3\int_0^{2\pi} \int_0^2 \left(r^3 + 2r\right) drd\theta$$

$$= 3\int_0^{2\pi} \left(\frac{r^4}{4} + r^2\right)\Bigg|_0^2 d\theta = 3 \cdot 8 \int_0^{2\pi} d\theta = 48\pi$$

Since the problem asked for the inward flux the answer is -48π